On the geometry of almost $\mathcal{S}$-manifolds

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Abstract

An f-structure on a manifold M is an endomorphism field φ satisfying $\varphi^3 + \varphi = 0$. We call an f-structure *regular* if the distribution $T = \ker\varphi$ is involutive and regular, in the sense of Palais. We show that when a regular f-structure on a compact manifold M is an almost $\mathcal{S}$-structure, it determines a torus fibration of M over a symplectic manifold. When $\operatorname{rank} T = 1$, this result reduces to the Boothby-Wang theorem. Unlike similar results for manifolds with $\mathcal{S}$-structure or $\mathcal{K}$-structure, we do not assume that the f-structure is normal. We also show that given an almost $\mathcal{S}$-structure, we obtain an associated Jacobi structure, as well as a notion of symplectization.

1 Introduction

Let (M, η) be a cooriented contact manifold. The Boothby-Wang theorem [4] tells us that if the Reeb field ξ corresponding to the contact form η is regular (in the sense of Palais [16]), then M is a prequantum circle bundle $\pi : M \to N$ over a symplectic manifold (N, ω), where $\pi^*\omega = -d\eta$, and η may be identified with the connection 1-form. Conversely, let M be a prequantum circle bundle over a symplectic manifold (N, ω) and let η be a connection 1-form. Given a choice of compatible almost complex structure J for ω, let $G(X, Y) = \omega(JX, Y)$ be the associated Riemannian metric on N, and let $\tilde{\pi}$ denote the horizontal lift of vector fields defined by η. We can then define an endomorphism field $\varphi \in \Gamma(M, \operatorname{End}(TM))$ by

$$\varphi X = \tilde{\pi} J \pi_* X,$$

and a Riemannian metric g by $g = \pi^*G + \eta\otimes\eta$. If we let ξ be the vertical vector field satisfying $\eta(\xi) = 1$, then (φ, ξ, η, g) defines a contact metric structure on M [2]. In particular, we note

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that φ is an f-structure on M. By construction, we have $\varphi^2 = -\operatorname{Id}_{TM} + \eta \otimes \xi$, from which it follows that $\varphi^3 + \varphi = 0$.

In [1, 3], Blair et al consider compact Riemannian manifolds equipped with a regular normal f-structure φ, and show that such manifolds are the total space of a principal torus bundle over a complex manifold N, and that in addition, N is a Kähler manifold if the fundamental 2-form of the f-structure is closed (that is, if M is a $\mathcal{K}$-manifold). Saenz argued in [17] that if this $\mathcal{K}$-structure is an $\mathcal{S}$-structure, then the symplectic form of the Kähler manifold N is integral.

While the results in [3, 17] provide us with a generalization of the Boothby-Wang theorem, the proofs in [3] (and by extension, the argument in [17]) rely in several places on the assumption that the f-structure φ is normal. Since this assumption is not required in the original Boothby-Wang theorem, it is natural to ask what can be said if this assumption is dropped for f-structures of higher corank. In this note, we use a theorem of Tanno [19] to show that if M is a compact almost $\mathcal{S}$-manifold, in the sense of [6], then M is a principal torus bundle over a symplectic manifold whose symplectic form is integral. (More precisely, the symplectic form will be a real multiple of an integral symplectic form.) Not surprisingly, this tells us that requiring φ to be normal is the same as demanding that the base of our torus bundle be Kähler.

This "generalized Boothby-Wang theorem" is one of a number of similarities between manifolds with almost $\mathcal{S}$-structure and contact manifolds. In the final section of this paper we demonstrate two more. First, there is a natural notion of symplectization: given an almost $\mathcal{S}$-manifold M, there is an open, conic, symplectic submanifold of T^*M whose base is M. Second, a choice of one-form (expressed in terms of the almost $\mathcal{S}$-structure) allows us to define a Jacobi bracket on the algebra of smooth functions on M, giving us in particular a notion of Hamiltonian vector field on manifolds with almost S-structure.

2 Preliminaries

2.1 Regular involutive distributions

Let $F \subset TM$ be an involutive distribution of rank k. We briefly recall the notion of a regular distribution in the sense of Palais, and refer the reader to [16] for the details. Roughly speaking, the involutive distribution F is *regular* if each point $p \in M$ has a coordinate neighbourhood $(U, x^1, \ldots . x^n)$ such that

$$\left\{ \left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^k}\right)_p \right\}$$

forms a basis for $F_p \subset T_pM$, and such that the integral submanifold of F through p intersects U in only one k-dimensional slice. When F is regular, the leaf space $\mathcal{F} = M/F$ is a smooth Hausdorff manifold, and the quotient mapping $\pi_F : M \to \mathcal{F}$ is smooth and closed. When M is compact and connected, the leaves of F are compact and isomorphic, and are the fibres of the smooth fibration $\pi_F : M \to \mathcal{F}$.

In particular, a vector field X on M is regular if each $p \in M$ has a neighbourhood U through which the integral curve of X through p passes only once. If M is compact, the

integral curves of a regular vector field are thus diffeomorphic to circles. Applying this fact to the Reeb vector field of a contact manifold gives part of the proof of the Boothby-Wang theorem.

2.2 f-structures

An *f-structure* on M is an endomorphism field $\varphi \in \Gamma(M, \operatorname{End} TM)$ such that

$$\varphi^3 + \varphi = 0. \tag{1}$$

Such structures were introduced by K. Yano in [21]; many of the facts regarding f structures are collected in the book [11]. By a result of Stong [18], every f-structure is of constant rank. If $\operatorname{rank}\varphi = \dim M$, then φ is an almost complex structure on M, while if $\operatorname{rank}\varphi = \dim M - 1$, then φ determines an almost contact structure on M.

It is easy to check that the operators $l = -\varphi^2$ and $m = \varphi^2 + \mathrm{Id}_{TM}$ are complementary projection operators; letting $E = l(TM) = \operatorname{im}\varphi$ and $T = m(TM) = \ker\varphi$, we obtain the splitting

$$TM = E \oplus T = \operatorname{im}\varphi \oplus \ker\varphi \tag{2}$$

of the tangent bundle. Since $(\varphi|_E)^2 = -\mathrm{Id}_E$, φ is necessarily of even rank. When the corank of φ is equal to one, the distribution T is automatically trivial and involutive. However, if $\operatorname{rank} T > 1$, this need not be the case, and one often makes additional simplifying assumptions about T. An f-structure such that T is trivial is called an *f-structure with parallelizable kernel* (or $f{\cdot}$pk-structure for short) in [6]. We will assume that an $f{\cdot}$pk-structure includes a choice of a trivializing frame $\{\xi_i\}$ and corresponding coframe $\{\eta^i\}$ for T^*, with

$$\eta^i(\xi_j) = \delta^i_j, \quad \varphi(\xi_i) = \eta^j \circ \varphi = 0, \quad \text{and} \quad \varphi^2 = -\mathrm{Id} + \sum \eta^i \otimes \xi_i.$$

(This is known as an *f-structure with complemented frames* in [1]; such a choice of frame and coframe always exists.) Given an $f{\cdot}$pk-structure, it is always possible [11] to find a Riemannian metric g that is compatible with (φ, ξ_i, η^j) in the sense that, for all $X, Y \in \Gamma(M, TM)$, we have

$$g(X, Y) = g(\varphi X, \varphi Y) + \sum_{i=1}^{k} \eta^i(X)\eta^i(Y). \tag{3}$$

Following [6], we will call the 4-tuple $(\varphi, \xi_i, \eta^j, g)$ a *metric $f{\cdot}$pk structure.* Given a metric $f{\cdot}$pk-structure $(\varphi, \xi_i, \eta^j, g)$, we can define the *fundamental 2-form* $\Phi_g \in \mathcal{A}^2(M)$ by

$$\Phi_g(X, Y) = g(\varphi X, Y). \tag{4}$$

Remark 2.1. Our definition of Φ_g is chosen to agree with our preferred sign conventions in symplectic geometry; however, many authors place φ in the second slot, so our convention here uses the opposite sign of that found for example in [3] and [6].

We will call an f-structure φ *regular* if the distribution $T = \ker\varphi$ is regular in the sense of Palais [16]. An $f{\cdot}$pk-structure is regular if the vector fields ξ_i are regular and independent.

An $f\cdot$pk-structure is called *normal* [1] if the tensor N defined by

$$N = [\varphi, \varphi] + \sum_{i=1}^{k} d\eta^i \otimes \xi_i, \tag{5}$$

vanishes identically. Here $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, which is given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

When φ is normal, the $+i$-eigenbundle of φ (extended by $\mathbb{C}$ linearity to $T_{\mathbb{C}}M$) defines a CR structure $E_{1,0} \subset T_{\mathbb{C}}M$. Regular normal f-structures are studied in [3], where it is proved that a compact manifold with regular normal f-structure is a principal torus bundle over a complex manifold N. If the fundamental 2-form Φ_g of a normal f-structure is closed, then the f-structure is called a $\mathcal{K}$-structure, and M a $\mathcal{K}$-manifold. For a compact regular $\mathcal{K}$-manifold M, the base N of the torus fibration is a Kähler manifold. A special case of a $\mathcal{K}$-manifold is an $\mathcal{S}$-manifold. On an $\mathcal{S}$ manifold there exist constants $\alpha^1, \ldots, \alpha^k$ such that $d\eta^i = -\alpha^i \Phi_g$ for $i = 1, \ldots, k$. Two commonly considered cases are the case $\alpha^i = 0$ for all i, and the case $\alpha^i = 1$ for all i. In the language of CR geometry, the former case is analogous to a "Levi-flat" CR manifold, while the latter defines an analogue of a strongly pseudoconvex CR manifold (typically, strongly pseudoconvex CR manifolds are assumed to be of "hypersurface type," meaning that the complementary distribution T has rank one; see [5]).

A refinement of the notion of $\mathcal{S}$-structure was introduced in [6]: a metric $f\cdot$pk-structure $(\varphi, \xi_i, \eta^j, g)$ which is not necessarily normal is called an *almost* $\mathcal{S}$-structure if $d\eta^i = -\Phi_g$ for each $i = 1, \ldots, k$. An f-structure φ is called CR-integrable in [6] if the $+i$-eigenbundle $E_{1,0} \subset T_{\mathbb{C}}M$ of φ is involutive (and hence, defines a CR structure). It is shown in [6] that an $f\cdot$pk-structure is CR-integrable if and only if the tensor N given by (5) satisfies $N(X, Y) = 0$ for all $X, Y \in \Gamma(M, E)$, where $E = \operatorname{im} \varphi$, whereas for a normal $f\cdot$pk-structure, N must vanish for all $X, Y \in \Gamma(M, TM)$. In [14] it is proved that a CR-integrable almost $\mathcal{S}$-manifold admits a canonical connection analogous to the Tanaka-Webster connection of a strongly pseudoconvex CR manifold. For the relationship between this connection and the $\overline{\partial}_b$ operator of the corresponding tangential Cauchy-Riemann complex, as well as an application of this relationship to defining an analogue of geometric quantization for almost $\mathcal{S}$-manifolds, see [7].

In this paper, we will define an almost $\mathcal{K}$-structure to be a metric $f\cdot$pk-structure for which $d\Phi_g = 0$, and we will define an almost $\mathcal{S}$-structure more generally to be an almost $\mathcal{K}$-structure such that $d\eta^i = -\alpha^i \Phi_g$ for constants $\alpha^i \in \mathbb{R}$, for $i = 1, \ldots, k$.

3 Properties of almost $\mathcal{K}$ and almost $\mathcal{S}$-structures

Let (φ, ξ_i, η^i) be an $f\cdot$pk-structure on a compact, connected manifold M. Let g be a Riemannian metric satisfying the compatibility condition (3), and let Φ_g denote the corresponding fundamental 2-form. Let $E = \operatorname{im} \varphi$, and $T = \ker \varphi$ denote the distribution spanned by the ξ_i. It's easy to check that the distributions E and T are orthogonal with respect to g, and that the restriction of Φ_g to $E \otimes E$ is nondegenerate, from which we have the following:

Lemma 3.1. *$X \in \Gamma(M,T)$ if and only if $\iota(X)\Phi_g = 0$.*

Proposition 3.2. *Let $(\varphi, \xi_i, \eta^i, g)$ be a metric $f\cdot pk$-structure. Then $T = \ker\varphi$ is involutive whenever $d\Phi_g = 0$.*

Proof. Let $X, Y \in \Gamma(M,T)$, and let $Z \in \Gamma(M,TM)$. Then, using Lemma 3.1 above, we have

$$\begin{aligned} d\Phi_g(X,Y,Z) &= X\cdot\Phi_g(Y,Z) + Y\cdot\Phi_g(Z,X) + Z\cdot\Phi_g(X,Y) \\ &\qquad - \Phi_g([X,Y],Z) - \Phi_g([Y,Z],X) - \Phi_g([Z,X],Y) \\ &= -\Phi_g([X,Y],Z). \end{aligned}$$

Therefore, if $d\Phi_g = 0$, then $\iota([X,Y])\Phi_g = 0$, and thus $[X,Y] \in \Gamma(M,T)$, which proves the proposition. □

Let us now suppose that $(\varphi, \xi_i, \eta^i, g)$ is an almost $\mathcal{S}$-structure, so that the 1-forms η^i satisfy $d\eta^i = -\alpha^i\Phi_g$ for constants α^i, some of which may be zero. The following results were proved in [6] in the case that $\alpha^i = 1$ for all i; we easily see that the results remain true in our more general setting:

Proposition 3.3. *If $(\varphi, \xi_i, \eta^j, g)$ is an almost $\mathcal{S}$-structure, then $\mathcal{L}(\xi_i)\xi_j = [\xi_i, \xi_j] = 0$ for all $i, j = 1, \ldots, k$.*

Proof. Since the fundamental 2-form Φ_g of an almost $\mathcal{S}$-structure is closed, the distribution T is involutive. Thus we may write $[\xi_i, \xi_j] = \sum c^a_{ij}\xi_a$. But for any $a, i, j \in \{1, \ldots, k\}$, we have

$$c^a_{ij} = \eta^a([\xi_i, \xi_j]) = \xi_i \cdot \eta^a(\xi_j) - \xi_j \cdot \eta^a(\xi_i) - d\eta^a(\xi_i, \xi_j) = \alpha^a\Phi_g(\xi_i, \xi_j) = 0.$$ □

Proposition 3.4. *If $(\varphi, \xi_i, \eta^j, g)$ is an almost $\mathcal{S}$-structure, then $\mathcal{L}(\xi_i)\eta^j = 0$ for all $i, j = i, \ldots, k$.*

Proof. We have $\mathcal{L}(\xi)\eta^j = d(\eta^j(\xi_i)) + \iota(\xi_i)d\eta^j = -\alpha^j(\iota(\xi_i)\Phi_g) = 0$. □

We remark that several other results from [6] hold in this more general setting, but they are not needed here. To conclude this section, we state a theorem due to Tanno [19]:

Theorem 3.5. *For a regular and proper vector field X on a manifold M, the following are equivalent:*

(i) The period function λ_X of X is constant.

(ii) There exists a 1-form η such that $\eta(X) = 1$ and $\mathcal{L}(X)\eta = 0$.

(iii) There exists a Riemannian metric g such that $g(X,X) = 1$ and $\mathcal{L}(X)g = 0$.

In the above theorem, the period function $\lambda_X : M \to \mathbb{R}$ is defined by

$$\lambda_X(p) = \inf\{t > 0 | \exp(tX)\cdot p = p\}. \tag{6}$$

If M is noncompact, the value $\lambda_X(p) = \infty$ is possible. Part (iii) of the above tells us that X is a unit Killing field for the metric g. Using this result, Tanno was able to give a simple proof (which is reproduced in [2]) of the Boothby-Wang theorem [4].

4 The structure of regular almost $\mathcal{S}$-manifolds

As noted above, from [3], a compact manifold with regular normal f-structure is a principal torus bundle over a complex manifold N, and N is Kähler if M is a $\mathcal{K}$-manifold. If M is an $\mathcal{S}$-manifold with $\Phi_g = -d\eta^i$ for each i, then by [17], the symplectic form on N is integral. We now dispense with the requirement that the f-structure on M be normal, and state a similar result for almost $\mathcal{S}$-manifolds.

Theorem 4.1. *Let M be a compact manifold of dimension $2n+k$ equipped with a regular almost $\mathcal{S}$-structure $(\varphi, \tilde{\xi}_i, \tilde{\eta}^i, \tilde{g})$ of rank $2n$. Then there exists an almost $\mathcal{S}$-structure $(\varphi, \xi_i, \eta^i, g)$ on M for which the vector fields $\xi_1, \ldots, \xi_k$ are the infinitesimal generators of a free and effective $\mathbb{T}^k$-action on M. Moreover, the quotient $N = M/\mathbb{T}^k$ is a smooth symplectic manifold of dimension $2n$, and if the α^i such that $d\tilde{\eta}^i = -\alpha^i \Phi_{\tilde{g}}$ are not all zero, then the symplectic form on N is a real multiple of an integral symplectic form.*

Proof. By assumption, the vector fields $\tilde{\xi}_1, \ldots, \tilde{\xi}_k$ are regular, independent and proper, and by Proposition 3.2, the distribution $T = \operatorname{span}\{\tilde{\xi}_1, \ldots, \tilde{\xi}_k\}$ is involutive. Thus, by the results of Palais, $N = M/T$ is a smooth manifold, and $\pi : M \to N$ is a smooth fibration whose fibres are the leaves of the distribution T. Since M is compact, the fibres are compact and isomorphic [16]. For each $i = 1, \ldots, k$, we have $\tilde{\eta}^i(\tilde{\xi}_i) = 1$ and $\mathcal{L}(\tilde{\xi}_i)\tilde{\eta}^i = 0$. Thus, by Theorem 3.5, the period functions $\lambda_i = \lambda_{\tilde{\xi}_i}$ are constant. We rescale by setting $\xi_i = \lambda_i \tilde{\xi}_i$ and $\eta^i = \frac{1}{\lambda_i}\tilde{\eta}^i$. We still have $\eta^i(\xi_j) = \delta^i_j$, and note that the associated metric g for which $(\varphi, \xi_i, \eta^i, g)$ is an almost $\mathcal{S}$-structure differs from $\tilde{g}$ only along T, so that $\Phi_g = \Phi_{\tilde{g}}$. Each ξ_i now has period 1, and since the vector fields ξ_i all commute, they are the generators of a free and effective $\mathbb{T}^k$-action on M. The argument for local triviality is the same as in [3], so we do not repeat it here. Thus, we have that M is a principal $\mathbb{T}^k$-bundle over $N = M/T$. The infinitesimal action of $\mathbb{R}^k$ is given by

$$X = (t^1, \ldots, t^k) \mapsto X_M = \sum t^i \xi_i,$$

from which we see that $\boldsymbol{\eta} = (\eta^1, \ldots, \eta^k)$ is a connection 1-form on M: we have $\iota(X_M)\eta = X$ and $\mathcal{L}(X_M)\eta = 0$ for all $X \in \mathbb{R}^k$.

Now, we note that the fundamental 2-form Φ_g is horizontal and invariant, since $\iota(X)\Phi_g = \mathcal{L}(X)\Phi_g = 0$ for all $X \in \Gamma(M, T)$, and thus there exists a 2-form Ω on N such that $\pi^*\Omega = \Phi_g$. Since $\pi^* d\Omega = d\Phi_g = 0$, Ω is closed, and since $\pi^*\Omega^n = \Phi_g^n \neq 0$, Ω is non-degenerate, and hence symplectic.

Finally, let us suppose that one of the α^i are non-zero; without loss of generality, let's say $\alpha^1 \neq 0$. By the same argument as above, the vector fields $\xi_2, \ldots, \xi_k$ generate a free $\mathbb{T}^{k-1}$-action on M, giving us a fibration $p : M \to P$. Now, since $\mathcal{L}(\xi_i)\xi_1 = \mathcal{L}(\xi_i)\eta^1 = 0$ for $i = 2, \ldots, k$, the vector field ξ_1 and 1-form η^1 are invariant under the $\mathbb{T}^{k-1}$-action. We can thus define a 1-form η on P by $\eta(X) = \eta^1(\tilde{p}X)$, where $\tilde{p}X$ denotes the horizontal lift of X with respect to the connection 1-form defined by $\eta^2, \ldots, \eta^k$, and a vector field ξ on P by $\xi = p_*\xi_1$. Note that $d\eta(X,Y) = d\eta^1(\tilde{p}X, \tilde{p}Y)$. We then have $\eta(\xi) = 1$, and $\mathcal{L}(\xi)\eta = \iota(\xi^1)d\eta^1 = 0$, so that Theorem 3.5 applies to the pair (η, ξ). It follows that ξ generates a free action of $S^1 = \mathbb{R}/\mathbb{Z}$ on P, giving us the $\mathbb{T}^1$-bundle structure $q : P \to N$. Since $\pi = q \circ p$, it follows

that

$$d\eta(X,Y) = d\eta^1(\tilde{p}X, \tilde{p}Y) = -\frac{\alpha^1}{\lambda_1}(\pi^*\Omega)(\tilde{p}X, \tilde{p}Y) = -\frac{\alpha^1}{\lambda_1} q^*\Omega(X,Y).$$

Thus, P is a Boothby-Wang fibration over $(N, \frac{\alpha^1}{\lambda_1}\Omega)$, from which it follows that the symplectic form $\frac{\alpha^1}{\lambda}\Omega$ must be integral (see [10]), and hence Ω is a real multiple of an integral symplectic form. $\square$

Remark 4.2. Note that since the last part of the argument is valid for any pair of nonzero constants α^i, α^j, from which it follows that for each i, j for which α^i and α^j are nonzero, we must have $\dfrac{\alpha^i}{\lambda_i} \cdot \dfrac{\lambda_j}{\alpha^j} \in \mathbb{Q}$.

Conversely, we have the following theorem:

Theorem 4.3. *Suppose that M is a principal $\mathbb{T}^k$-bundle over a symplectic manifold (N, ω), equipped with connection 1-form $\boldsymbol{\eta} = (\eta^1, \ldots, \eta^k)$ such that there exist constants $\alpha^1, \ldots, \alpha^k$ for which $d\eta^i = -\alpha^i \pi^*\omega$. Then M admits an almost $\mathcal{S}$-structure.*

Proof. The proof is essentially the same as the proof given in [1] when N is Kähler, if we omit the proof of normality. Given a choice of compatible almost complex structure J and associated metric G, we can define an f-structure φ by $\varphi X = \tilde{\pi} J \pi_* X$, where $\tilde{\pi}$ denotes the horizontal lift with respect to $\boldsymbol{\eta}$. If we let $\xi_1, \ldots, \xi_k$ denote vertical vectors such that $\eta^i(\xi_j) = \delta^i_j$, and define the metric g by

$$g(X,Y) = \pi^* G(X,Y) + \sum \eta^i(X)\eta^i(Y),$$

then it's straightforward to check that the data $(\varphi, \xi_i, \eta^j, g)$ defines an almost $\mathcal{S}$-structure on M. (Note that $\Phi_g = \pi^*\omega$, so that $d\eta^i = -\alpha^i \Phi_g$.) $\square$

Remark 4.4. We can also use the results of Tanno [19] to show that the vector fields $\xi_1, \ldots, \xi_k$ of an almost $\mathcal{S}$-structure are Killing. Let $\tilde{\pi}$ denote the horizontal lift defined by $\boldsymbol{\eta}$. Then we can define a Riemannian metric G on N by $G(X,Y) = g(\tilde{\pi}X, \tilde{\pi}Y)$ for any $X, Y \in \Gamma(N, TN)$, where g is the metric of the almost $\mathcal{S}$-structure on M. It follows that $g = \pi^*G + \sum \eta^i \otimes \eta^i$, whence $g(\xi_i, \xi_i) = 1$ and $\mathcal{L}(\xi_i)g = 0$ for $i = 1\ldots,k$. Moreover, the endomorphism field $J \in \Gamma(N, \mathrm{End}(TN))$ defined by $JX = \pi_*\varphi\tilde{\pi}X$ is easily seen to be an almost complex structure on N that is compatible with G, and the symplectic form Ω then satisfies $\Omega(X,Y) = G(X, JY)$.

Remark 4.5. If M is only an almost $\mathcal{K}$-manifold, it is not clear that we can expect any analogous result to hold, since the proof in [3] for a $\mathcal{K}$-manifold does not work without normality, and Tanno's theorem cannot be applied if $\mathcal{L}(\xi_i)\eta^j \neq 0$ for all i, j, and this need not hold if $d\eta^j$ is not a multiple of Φ_g.

Remark 4.6. If M is noncompact, then as noted below the statement of Tanno's theorem, the period λ_i of one of the ξ_i could be infinite, in which case ξ_i generates an $\mathbb{R}$-action on M instead of an S^1-action.

5 Symplectization and Jacobi structures

We conclude this paper with a discussion of the relationship between almost $\mathcal{S}$-structures and related geometries intended to reinforce the view that almost $\mathcal{S}$-structures deserve to be viewed as higher corank analogues of contact structures. (However, see also [20] for the notion of k-contact structures, which, from the point of view of Heisenberg calculus, are also deserving of the title of higher corank contact structure. From this perspective, almost $\mathcal{S}$-structures are perhaps more analogous to contact metric structures, or even strongly pseudoconvex CR structures, although they are not CR-integrable in general.)

Recall that a stable complex structure on a manifold M is a complex structure defined on the fibres of $TM \oplus \mathbb{R}^k$ for some k. Given an $f{\cdot}$pk-structure (φ, ξ_i, η^j) on M, we obtain a stable complex structure $J \in \Gamma(M, \mathrm{End}(TM \oplus \mathbb{R}^k))$ by setting $JX = \varphi X$ for $X \in \Gamma(M, E)$, and defining $J\xi_i = \tau_i$ and $J\tau_i = -\xi_i$, where $\tau_1, \ldots, \tau_k$ is a basis for $\mathbb{R}^k$. As explained in [8], a stable complex structure determines a Spin^c-structure on M.

Alternatively, (and with some abuse of notation), we can think of the above complex structure on each fibre $T_xM \times \mathbb{R}^k$ as coming from an almost complex structure on $M \times \mathbb{R}^k$ obtained from to the f-structure φ. With this point of view, we note that it is possible to define a "symplectization" analogous to the symplectization of a cooriented contact manifold, provided that our $f{\cdot}$pk-structure is an almost $\mathcal{S}$-structure, with at least one of the α^j (such that $d\eta^j = -\alpha^j\Phi_g$) nonzero. As above, we let $TM = E \oplus T$ denote the splitting of the tangent bundle determined by the f-structure, and let $E^0 \cong T^* = \mathrm{span}\{\eta^i\} \cong M \times \mathbb{R}^k$ denote the annihilator of E. It is then possible to find an open connected symplectic submanifold E^0_+ of T^*M whose tangent bundle is $T_xM \times \mathbb{R}^k$. For concreteness, let us use the identification $E^0 \cong M \times \mathbb{R}^k$, and with respect to coordinates $(x, t_1, \ldots, t_k)$, let

$$\alpha = \sum_{i=1}^{k} t_i \eta^i,$$

and define $\omega = -d\alpha$. (We are abusing notation here slightly; technically we should write $\pi^*\eta^i$ in place of η^i, where $\pi : M \times \mathbb{R}^k \to M$ is the projection onto the first factor.) Using the fact that $d\eta^i = -\alpha^i \Phi_g$ for each i, we have

$$\omega = \sum \eta^j \wedge dt_j + \left(\sum t_j \alpha^j\right) \Phi_g.$$

Define $\tau \in C^\infty(E^0)$ to be the function given in coordinates by $\tau = \sum \alpha^j t_j$. Note that since $\eta^i \wedge \eta^i = dt_i \wedge dt_i = 0$, we have

$$\left(\sum_{i=1}^{k} \eta^j \wedge dt_j\right)^k = k!\, \eta^1 \wedge dt_1 \wedge \cdots \wedge \eta^k \wedge dt_k.$$

We also note that $\Phi_g^m = 0$ for $m > n$. Thus, using the binomial theorem, we find that the top-degree form ω^{n+k} has only one nonzero term; namely,

$$\omega^{n+k} = \frac{(n+k)!}{n!} \eta^1 \wedge dt_1 \cdots \wedge \eta^k \wedge dt_k \wedge (\tau \Phi_g)^n.$$

Thus, ω^{n+k} is a volume form on the open subset E^0_+ of E^0 defined by $\tau > 0$, and hence ω is a symplectic form on E^0_+.

Next, we will show that for certain choices of section $\eta \in \Gamma(M, E^0)$ we obtain a Jacobi structure on M defined in a manner analogous to the Jacobi structure associated to a choice of contact form on a contact manifold. We recall that a Jacobi structure on M is given by a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ such that for any $f, g \in C^\infty(M)$ the support of $\{f, g\}$ is contained in the intersection of the supports of f and g. Jacobi structures were introduced independently by Kirillov [9] and Lichnerowicz [13]; a good introduction can be found in [15].

Again, we assume M is equipped with an almost $\mathcal{S}$-structure with the constants α^j such that $d\eta^j = -\alpha^j\Phi_g$ not all zero. Our first goal is to define a notion of a Hamiltonian vector field X_f associated to each function $f \in C^\infty(M)$. To begin with, let $\xi = \sum b^j\xi_j$ be an arbitrary section of $T = \ker\varphi$, and let $\eta = \sum c_j\eta^j$ be an arbitrary section of $E^0 \cong T^*$. We will narrow down the possibilities for ξ and η as we consider the properties we wish the vector fields X_f to satisfy. The idea is to generalize the approach used to define Hamiltonian vector fields on a contact manifold (M, η). Recall that on manifold equipped with a contact form η, where we define $\Phi = -d\eta$, the Reeb vector field ξ is defined by $\iota(\xi)\eta = 1$ and $\iota(\xi)\Phi = 0$. A contact Hamiltonian vector field X_f satisfies the equations $\iota(X_f)\eta = f$ and $\iota(X_f)\Phi = df - (\xi\cdot f)\eta$. Lichnerowicz showed in [12] that these are the necessary and sufficient conditions for each X_f to be an infinitesimal symmetry of the contact structure: it follows that for each $f \in C^\infty(M)$, $\mathcal{L}(X_f)\eta = (\xi \cdot f)\eta$.

We wish to impose similar conditions on ξ, η and (the yet to be defined) X_f in the case of almost $\mathcal{S}$-manifolds. We already know that $\iota(\xi)\Phi_g = 0$, by Lemma 3.1, so we begin by adding the requirement that $\eta(\xi) = \sum b^jc_j = 1$. Next, we give our definition of a Hamiltonian vector field:

Definition 5.1. *Let η and ξ be as above. For any $f \in C^\infty(M)$, we define the* Hamiltonian vector field *associated to f by the equations*

$$\iota(X_f)\eta^j = \alpha^j f, \text{for } j = 1, \ldots, k, \tag{7}$$

$$\iota(X_f)\Phi_g = df - (\xi \cdot f)\eta. \tag{8}$$

Remark 5.2. Note that the above equations uniquely define X_f, by the nondegeneracy of the restriction of Φ to $E = \operatorname{im}\varphi$. The constants α^j are the same ones such that $d\eta^j = -\alpha^j\Phi_g$. One can check that if we began with a^j in place of the α^j, we would be forced to take $a^j = \alpha^j$ for consistency reasons. (In particular this will be necessary if the bracket we define below is to be a Lie bracket.) Moreover, this gives us the identity

$$\mathcal{L}(X_f)\eta^j = \alpha^j(\xi \cdot f)\eta$$

for each $j = 1, \ldots, k$; we would otherwise have an unwanted term of the form $(a^j - \alpha^j)df$. Note that on the right-hand side of the above equation we have η and not η^j; this is unavoidable with our definition of X_f.

We can fix the coefficients of ξ by requiring that ξ be the Hamiltonian vector field associated to the constant function 1, as is standard for Jacobi structures (see [15]). It is

easy to see that (7) then immediately forces us to take $\xi = \sum \alpha^j \xi_j$; that is, the coefficients b^j are equal the constants α^j. Thus, ξ is essentially determined by the almost $\mathcal{S}$ structure, although η is constrained only by the condition $\eta(\xi) = 1$, so the Jacobi structure we define below cannot be considered entirely canonical (as one might expect). From the requirement that $\eta(\xi) = 1$ it follows that for each $f \in C^\infty(M)$, we have

$$\mathcal{L}(X_f)\eta = \sum c_j \mathcal{L}(X_f)\eta^j = \sum c_j \alpha^j (\xi \cdot f)\eta = (\xi \cdot f)\eta,$$

again in analogy with the contact case. Note that the normalization $\eta(\xi) = 1$ also implies that $d\eta = -\Phi_g$. We are now ready to define our bracket on $C^\infty(M)$.

Definition 5.3. *Let M be a manifold with almost $\mathcal{S}$-structure, with constants α^j not all zero. Let $\xi = \sum \alpha^j \xi_j$, and let η be a section of E^0 such that $\eta(\xi) = 1$. We then define a bracket on $C^\infty(M)$ by*

$$\{f, g\} = \iota([X_f, X_g])\eta. \tag{9}$$

The bracket is clearly antisymmetric, and one checks (using the identity $\iota([X, Y]) = [\mathcal{L}(X), \iota(Y)]$) that

$$\{f, g\} = X_f \cdot g - X_g \cdot f + \Phi_g(X_f, X_g) = X_f \cdot g - (\xi \cdot f)g.$$

Note that since the definition of the Hamiltonian vector fields depended on the choice of η, the bracket depends on η, even though η no longer appears explicitly in either of the above expressions for the bracket. From the latter equality we see that the support of $\{f, g\}$ is contained in the support of g, and by antisymmetry it must be contained in the support of f as well. Thus, the bracket given by (9) is a Jacobi bracket provided we can verify the Jacobi identity. Since the Jacobi identity is valid for the Lie bracket on vector fields, it suffices to prove the following:

Proposition 5.4. *Let $\{f, g\}$ be the bracket on $C^\infty(M)$ given by* (9)*. Then the vector field $X_{\{f,g\}}$ corresponding to the function $\{f, g\}$ is given by $X_{\{f,g\}} = [X_f, X_g]$.*

Lemma 5.5. *For each $i = 1, \ldots, k$, we have $[\xi_i, X_f] = X_{\xi_i \cdot f}$.*

Proof. From Propositions 3.3 and 3.4, we know that $[\xi_i, \xi_j] = 0$ and $\mathcal{L}(\xi_i)\eta^j = 0$ for any $i, j \in \{1, \ldots, k\}$; from the latter it follows easily that $\mathcal{L}(\xi_i)\Phi_g = 0$ as well. The result then follows from the uniqueness of Hamiltonian vector fields, since

$$\iota([\xi_i, X_f])\eta^j = [\mathcal{L}(\xi_i), \iota(X_f)]\eta^j = \alpha^j \xi_i \cdot f,$$

and

$$\iota([\xi_i, X_f])\Phi_g = \mathcal{L}(\xi_i)(df - (\xi \cdot f)\eta) = d(\xi_i \cdot f) - (\xi \cdot (\xi_i \cdot f))\eta.$$ □

Lemma 5.6. *For each $i = 1, \ldots, k$, we have $\xi_i \cdot \{f, g\} = \{\xi_i \cdot f, g\} + \{f, \xi_i \cdot g\}$.*

Proof. We have, using Lemma 5.5 and the fact that $[\xi_i, \xi] = 0$ in the second line,

$$\begin{aligned}
\xi_i \cdot \{f, g\} &= \xi_i \cdot (X_f \cdot g) - \xi_i \cdot ((\xi \cdot f)g) \\
&= X_f \cdot (\xi_i \cdot g) - (\xi \cdot f)(\xi_i \cdot g) + X_{\xi_i \cdot f} \cdot g - \xi(\xi_i \cdot f)g \\
&= \{f, \xi_i \cdot g\} + \{\xi_i \cdot f, g\}.
\end{aligned}$$ □

Proof of Proposition 5.4. We need to show that $\iota([X_f, X_g])\eta^j = \alpha^j\{f,g\}$ for each $j = 1, \dots k$, and that $\iota([X_f, X_g])\Phi = d\{f,g\} - (\xi \cdot \{f,g\})\eta$. First, since $\iota(X_g)\eta = \sum c_j\alpha^j g = g$, we have

$$\begin{aligned}\iota([X_f, X_g])\eta^j &= \mathcal{L}(X_f)\eta^j(X_g) - \iota(X_g)\mathcal{L}(X_f)\eta^j\\ &= \alpha^j X_f \cdot g - \iota(X_g)(\alpha^j(\xi \cdot f)\eta = \alpha^j\{f,g\}.\end{aligned}$$

From Lemma 5.6, we have $\xi \cdot \{f,g\} = \{f, \xi \cdot g\} - \{g, \xi \cdot f\} = X_f \cdot (\xi \cdot g) - X_g \cdot (\xi \cdot f)$, and thus,

$$\begin{aligned}\iota([X_f, X_g])\Phi_g &= \mathcal{L}(X_f)(dg - (\xi \cdot g)\eta) - \iota(X_g)(-d(\xi \cdot f) \wedge \eta + (\xi \cdot f)\Phi_g)\\ &= d(X_f \cdot g) - X_f \cdot (\xi \cdot g) - (\xi \cdot g)(\xi \cdot f)\eta + X_g \cdot (\xi \cdot f)\eta\\ &\quad - gd(\xi \cdot f) - (\xi \cdot f)(dg - (\xi \cdot g)\eta)\\ &= d(X_f \cdot g - (\xi \cdot f)g) - (X_f \cdot (\xi \cdot g) - X_g \cdot (\xi \cdot f))\eta\\ &= d\{f,g\} - \xi \cdot \{f,g\}\eta.\end{aligned}$$

□

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